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STABILITY OF N-TH ORDER AND DEGREE OF STABILITY

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1. Let us consider the system

$$y' = f(x, y) \tag{1.1}$$

where

$$y = (y_1, \ldots, y_k), \quad y' = (y_1', \ldots, y_k')$$

$$f(x, y) = (f(x, y), \ldots, f_k(x, y))$$

Definition 1. Let the column vectors y be a solution of the system [1] when $x \to \infty$, stable in the Liapunov sense. We shall call y the nth order stable solution if for each solution u of (1.1) for which

$$\| u - y \| < \varepsilon, \quad x_0 \leqslant x < \infty$$

there exists a positive number A such that the inequalities

$$\| u^{(m)} - y^{(m)} \| < A\varepsilon, \ \ x_0 \leqslant x < \infty, \ m = 1, \ldots, n \tag{1.2}$$

hold.

Definition 2. We shall say that y is a nth order asymptotically stable solution of (1.1) when it is nth order stable and the conditions

$$\lim_{x \to \infty} \| u^{(m)} - y^{(m)} \| = 0 \qquad (m = 0, 1, \ldots, n) \tag{1.3}$$

hold.

Example. Let us consider a rectilinear motion of a particle under the attraction of an immovable center, dtrectly proportional to the distance. The equation of motion is

$$mx^{\cdot\cdot} = -k^2 mx \tag{1.4}$$

and the general law of motion is given by

$$x = A \cos kt + B \sin kt$$

Clearly, $x = 0$ is a stable solution of (1.4). For a given ε and t_0 let us chose $|A| < \varepsilon/2$ and $|B| < \varepsilon/2$. Then $|x^{\cdot}| \leqslant |k| (|A| + |B|) \leqslant |k| \varepsilon, \ t_0 \leqslant t, \ |x^{\cdot\cdot}| < k^2\varepsilon$, etc. Therefore $x = 0$ is an nth order stable solution.

Theorem 1. Let $y = 0$ be an nth order asymptotically stable solution of (1.1), with $f(x, 0) \equiv 0$. Let u be a solution of (1.1) satisfying the conditions (1.3) and

$$\lim_{x \to \infty} u_{(x)}^{(m)} = u_{(\infty)}^{(m)} \qquad (m = 0, 1, \ldots, n) \tag{1.5}$$

Under these assumptions an nth order tangential contact exists at the point $y = 0$ between the solutions u themselves, and between the solutions $y = 0$.

Proof. From the condition of the nth order asymptotic stability and (1.5) we have

$$u_{(\infty)}^{(m)} = 0 \quad (m = 0, 1, \ldots, n)$$

and the proof of Theorem 1 follows.

Let us consider the following system of integro-differential equations:

$$y' = f(x, y, z), \quad z = \int_a^b K(x, t, y(t))\, dt \quad (a \leqslant b) \tag{1.6}$$

where we have the column vectors

$$z = (z_1, \ldots, z_k), \quad K = (K_1, \ldots, K_k)$$

Theorem 2. Let f satisfy the Lipshitz condition in y with the constant $B \geqslant 0$, and in z with the constant $C \geqslant 0$. Let K satisfy the Lipshitz condition in y with the constant $D \geqslant 0$. Then, if y is a stable or an asymptotically stable solution of (1.6), then it is a first order stable or asymptotically stable solution, respectively.

Proof. Let u be a solution of (1.6) for which

$$\| u - y \| < \varepsilon, \quad x_0 \leqslant x < \infty$$

Clearly

$$\| u' - y' \| = \left\| f\left[x, u, \int_a^b K(x, t, u(t))\, dt\right] - f\left[x, y, \int_a^b K(x, t, y(t))\, dt\right] \right\| \leqslant \tag{1.7}$$

$$B \| u - y \| + C \int_a^b D \| u - y \| dt < [B + CD(b - a)]\, \varepsilon$$

From Definition 1 it follows that y is a first order stable solution. If y is an asymptotically stable solution of (1.6), then $\lim \| u - y \| = 0$, as $x \to \infty$ for each u satisfying the condition

$$\| u(x_0) - y(x_0) \| < \delta(x_0, \varepsilon)$$

Then from (1.7) we obtain

$$\lim_{x\to\infty} \| u' - y' \| \leqslant \lim_{x\to\infty} \| u - y \| + CD \int_a^b \lim_{x\to\infty} \| u - y \| dt = 0$$

from which it follows that y is a first order asymptotically stable solution.

Let us consider the nonlinear system (where $\mathbf{A}(x)$ is a matrix)

$$y' = \mathbf{A}(x)\, y + f(x, y), \quad f(x, 0) \equiv 0 \tag{1.8}$$

Theorem 3. Let $\| \mathbf{A}(x) \| \leqslant A$ and let

$$\| f(x, y) \leqslant | \varphi(x) \| y \|^m, \quad | \varphi | \leqslant B \quad (m > 0, \; a \leqslant x < \infty)$$

where A and B are positive constants. Then if $y = 0$ is a stable or asymptotically stable solution of (1.8), then it is a first order stable or asymptotically stable solution, respectively.

Proof. Let y be a solution for which $\| y \| < \varepsilon$ for $x_0 \leqslant x < \infty$. Then

$$\| y' \| \leqslant \| A(x) \| \| y \| + \| \varphi \| \| y \|^m < \varepsilon [A + B\varepsilon^{m-1}]$$

It follows that $y = 0$ is a first order stable solution. If $y = 0$ is an asymptotically stable solution of (1.8), then $\lim y = 0$ as $x \to \infty$ and consequently $\lim y' = 0$. Thus $y = 0$ is a first order asymptotically stable solution.

Let us consider the linear system

$$y' = \mathbf{A}(x)\, y + \int_a^b \mathbf{K}(x, t)\, y(t)\, dt + f(x), \quad [f(x) = (f_1, \ldots, f_k)] \tag{1.9}$$

where $\mathbf{A}(x)$ and $\mathbf{K}(x, t)$ are matrices and $f(x)$ is a column vector.

Theorem 4. Let the system (1.9) have a definite, n-times differentiable solution in the interval $a \leqslant x < \infty$. Let the matrices A, K and f be ($n-1$)-times differentiable and bounded and let their ($n-1$)-th derivatives also be bounded

$$\| A^{(r)} \| \leqslant M_r, \quad \| K^{(r)} \| \leqslant N_r \quad (r = 0, 1, \ldots, n-1) \tag{1.10}$$

Then if y is a stable or an asymptotically stable solution of (1.9), it is an nth order stable or an asymptotically stable solution, respectively.

Proof. Let u be a solution for which

$$\| u - y \| < \varepsilon, \quad x_0 \leqslant x < \infty$$

It is clear that

$$\| u' - y' \| \leqslant \| \mathbf{A} \| \| u - y \| + \int_a^b \| \mathbf{K} \| \| u - y \| dt < \varepsilon \left[M_0 + N_0 (b - a) \right] = A_1 \varepsilon \tag{1.11}$$

Differentiating (1.9) and taking the condition (1.10) into account, we obtain

$$\| u^{(n)} - y^{(n)} \| < A_n \varepsilon \tag{1.12}$$

where $A_1, \ldots, A_n$ are positive constants of obvious origin. From this it follows that y is a nth order stable solution. Let y be an asymptotically stable solution of (1.9). Then

$$\lim_{x \to 0} \| u - y \| = 0 \quad \text{for} \quad \| u(x_0) - y(x_0) \| < \delta (x_0, \varepsilon)$$

From (1.11) and (1.12) we see that $\lim_{x \to \infty} \| u^{(r)} - y^{(r)} \| = 0, \quad r = 1, \ldots, n$. Therefore y is an nth order asymptotically stable solution.

Corollary. If a system of linear differential equations with constant coefficients has an n-tuply differentiable, stable or asymptotically stable solution in the interval $a \leqslant x \leqslant \infty$, then this solution is an nth order stable or asymptotically stable solution, respectively.

Consider the system

$$y' = f(x, y) \tag{1.13}$$

Theorem 5. Let $f(x, y)$ satisfy the Lipshitz condition in y with the constant A. Let also f_x' and f_y' exist, $\| f_y' \| < B$, and f_x' satisfy the Lipshitz condition in y with the constant C in the region $D \{a \leqslant x < \infty, \| y \| < H\}$ in which (1.13) has a solution. Then, if y is a stable or asymptotically stable solution of (1.13), then this solution is a second order stable or asymptotically stable solution, respectively.

Proof. Let u be a solution for which $\| u - y \| < \varepsilon, \quad x_0 \leqslant x < \infty$. Then

$$\begin{aligned} &\| u' - y' \| \leqslant \| f(x, u) - f(x, y) \| < A\varepsilon \\ &\| u'' - y'' \| \leqslant \| f_x'(x, u) - f_x'(x, y) \| + \\ &\qquad \| f_y'(x, u) u' - f_y'(x, y) y' \| < \varepsilon [C + AB] \end{aligned} \tag{1.14}$$

From (1.14) we see that y is a second order stable solution. Let y be an asymptotically stable solution of (1.13). Then

$$\lim \| u - y \| = 0 \quad \text{for} \quad x \to \infty$$

and from (1.14) follows

$$\lim \| u' - y' \| = 0, \quad \lim \| u'' - y'' \| = 0$$

i.e. y is a second order asymptotically stable solution.

2. Let $\varphi(t)$ be a continuous function monotonously increasing and differentiable in the interval (a, ∞), for which the conditions

$$\varphi \geqslant 1, \quad a \leqslant t < \infty, \quad \lim_{t\to\infty} \varphi = b \geqslant 1 \tag{2.1}$$

hold.

Definition 3. We say that a solution $\eta(t)$ of (1.1) is stable for $t \to \infty$ and of degree $n \geqslant 0$ with respect to the function φ the properties of which are given by (2.1), if for any $\varepsilon > 0$ and $t_0 \in (a, \infty)$ there exists $\delta = \delta(\varepsilon, t_0)$ such that every solution of the system (1.1) defined for $a \leqslant t < \infty$ and satisfying the condition

$$\| \varphi^n(t_0)\, y(t_0) - \eta(t_0) \| < \delta$$

also satisfies the condition

$$\| \varphi^n(t)\, y(t) - \eta(t) \| < \varepsilon, \quad t_0 \leqslant t < \infty \tag{2.2}$$

Here we call the number $n \geqslant 0$ the degree of stability of the solution η with respect to φ. If in addition $\lim_{t\to\infty} \| \varphi_y{}^n - \eta \| = 0$, then the solution η is asymptotically stable and of degree n with respect to φ. If $\delta = \delta(\varepsilon)$, we say that the stability is uniform.

Corollaries. If $\varphi \equiv 1$ in (a, ∞), then $y = 0$ is Liapunov stable. If $y = 0$ is a stable solution of (1.1) of degree $n \geqslant 0$ with respect to φ, then it is stable in the Liapunov sense. If $y = 0$ is a stable solution of (1.1) of degree $n > 0$ with respect to φ and $\lim_{t\to\infty} \varphi = \infty$, $n = k + r$, $k \geqslant 0$ and $r > 0$, then $y = 0$ is an asymptotically stable solution of degree k with respect to φ .

Theorem 6. Let in the system (1.1)

$$f(t, y) \in C_{t,y}^{(0,1)}(z), \quad z = \{a \leqslant t < \infty,\ \| y \| < H\}$$

$$\{z_0 = [a \leqslant t < \infty,\ \| y \| < h < H] \subset z\}, \quad f(t, 0) \equiv 0$$

If for the system (1.1) there exsists a positive definite scalar function

$$V(t, \varphi^n y) \in C_{t,y}^{(1,1)}(z_0 \subset z), \quad n \geqslant 0$$

admitting a negative derivative with respect to time (when y' in V' is replaced by the corresponding quantity according to (1.1)) and if there also exists a negative function $W(\varphi^n y) \in z_0$ and continuous in z_0 , for which

$$V(t, \varphi^n y) \geqslant W(\varphi^n y) > 0 \tag{2.3}$$

$$V(t, 0) = W(0) = 0 \quad \text{for} \quad y \neq 0$$

then the solution $y \equiv 0$ $(a \leqslant t < \infty)$ is a stable solution of degree n with respect to φ as $t \to \infty$, the properties of φ being defined by (2.1).

Theorem 7. Let the positive definite functions

$$V(t, \varphi^n y) \in C_{t,y}^{(1,1)}(z_0), \quad W(\varphi_y{}^n) \in C_{t,y}(z_0), \quad n \geqslant 0$$

for which the conditions (2.3) hold, exist for the system (1.1). Let V have an infinitely small upper limit as $y \to 0$ and let its derivative with respect to time be negative (when y' in V' is replaced by the corresponding quantity according to (1.1)). Let $W_1(\varphi^n y)$ be a positive function continuous in z_0, $V' > W_1(\varphi^n y)$, with the properties of φ given by (2.1). Then $y = 0$ is an asymptotically stable solution of (1.1) of degree n with respect to φ.

Theorems 6 and 7 follow from the theorems of Liapunov [2].

Theorem 8. If $u = 0$ is a Liapunov stable or asymptotically stable solution of the system

$$\left(\frac{u}{\varphi^n(t)}\right)' = f\left(t, \frac{u}{\varphi^n(t)}\right), \quad f(t,0) \equiv 0, \quad n \geqslant 0 \tag{2.4}$$

then $y = 0$ is, respectively, a stable or asymptotically stable solution of

$$y' = f(t, y) \tag{2.5}$$

of degree n with respect to φ. Conversely, if $y = 0$ is a stable or asymptotically stable solution of (2.5) of degree n with respect to φ, then $u = 0$ is a Liapunov stable or asymptotically stable solution of (2.4), respectively.

Proof. Let $u = 0$ be a stable solution of (2.4). Then $\| u \| < \varepsilon$ when $t_0 \leqslant t$, if $\| u(t_0) \| < \delta < \varepsilon$. But $u = \varphi^n y$, consequently $\| u \| = \| \varphi^n(t)\, y(t) \| < \delta$ when $t_0 < t$ and $\| \varphi^n(t_0)\, y(t_0) \| < \delta < \varepsilon$. Therefore $y = 0$ is a stable solution of (2.5) of degree n with respect to φ. If $\lim \| u \| = 0$ for $t \to \infty$, then $\lim \| \varphi^n y \| = 0$ as well, i.e. if $u = 0$ is an asymptotically stable solution of (2.4), then $y = 0$ is an asymptotically stable solution of (2.5) of degree n with respect to φ. The converse is obviously also true.

Theorem 9. From the exponential stability of the solution $y = 0$ of the system (1.1), i.e. from

$$\| y \| \leqslant N \| y(t_0) \| e^{-n(t-t_0)}$$

(N and n are positive constants) it follows that $y = 0$ is a stable solution of degree n with respect to e^{t-t_0}.

Proof. It is clear that

$$e^{n(t-t_0)} \| y \| \leqslant N \| (t_0) \| < \varepsilon \; (t_0 \leqslant t < \infty)$$

provided we choose $\| y(t_0) \| < \varepsilon / N$, and this proves the theorem. It is also obvious that if $\beta > 0$ and $\beta < n$, then $y = 0$ is an asymptotically stable solution of degree β with respect to e^{t-t_0}. It can be shown that if $y = 0$ is an exponentially stable solution of (1.1), then it is asymptotically stable of any degree with respect to t.

Definition 4. The solution $\eta(t)$ $(t_0 \leqslant t < \infty)$ of the system (1.1) shall be called an orbitally stable solution of degree n, with respect to the function $\varphi(t)$, the properties of which are given in (2.1), for $t \to \infty$, provided that for an arbitrarily small $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\| \varphi^n(t_0)\, y(t_0) - \eta(t_0) \| < \delta$, then

$$\rho\,(\varphi^n(t)\, y(t),\; L_0^+) < \varepsilon, \quad t_0 \leqslant t \tag{2.6}$$

If in addition

$$\lim_{t \to \infty} \rho\,(\varphi^n(t)\, y(t),\; L_0^+) = 0$$

then the solution η will be called orbitally asymptotically stable of degree n with respect to φ. Here L_0^+ is the positive half of the trajectory of the solution η (see [3]).

Theorem 8 remains valid for the orbital stability, and it can be proved in an identical manner after replacing the notion of stability by the notion of orbital stability and (2.2) by (2.6).

The definition of stability can be generalized by assuming that the condition (2.2) holds for the derivatives, i.e.

$$\| \varphi^n(t)\, u^{(m)} - y^{(m)} \| < A\varepsilon, \quad m = 0, 1, \ldots, p$$

and a number of theorems can be proved.

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ON THE PROBLEM OF IMPRESSING A THIN RIGID BODY INTO A MEDIUM WITH HARDENING

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The boundary separating the plastic and the rigid domains is determined on the basis of the linearized solution [1] of the problem of impressing a thin rigid body into a plastic medium possessing translational hardening. The case of a solid in the shape of a wedge is considered. In particular, the solution of the problem of impressing a thin wedge into an ideally plastic half-space is obtained when hardening is neglected; a comparison is made with the known solution of Hill, Lee and Tupper [2].

1. Considering a plastic material to be under plane strain conditions, and directing the coordinate axes as shown in Fig. 1a, let us write the equation of the solid surface as

$$y = \delta f(x), \quad f(0) = 0, \qquad F_2(0) = 0 \ (F_i \equiv d^i f / dx^i) \tag{1.1}$$

where δ is a small dimensionless parameter, and f is a sufficiently smooth function. The material occupies the half-space $x \leqslant 0$ at the initial instant. Reversing the motion, let us consider the solid fixed and the medium to move translationally upward along the x-axis at some constant velocity.

Let us henceforth use the variables

$$\eta = x - y, \quad \xi = x + y \tag{1.2}$$

in addition to the variables x, y.

The linearized solution of the problems has been found in [1]. It follows therefrom that the plastic domain AOB (Fig. 2) consists of two zones: OBC $(0 \leqslant \xi \leqslant h)$ and ABC $(h \leqslant \xi \leqslant 2h)$. The stresses on the line BC $(\xi = h)$ are continuous. The equation of the buckling surface of the plastic material is [1]

$$x - h = \delta f(h - y) \tag{1.3}$$

In a zero approximation the boundary separating the plastic and rigid domains is defined by the equation $x - y = 0$ (Fig. 1b), which in the η, ξ variables has the following form:

$$\eta(\xi) = 0 \tag{1.4}$$